An Example of QR Decomposition

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Compute the QR decomposition of

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

This example is adapted from the book, "*Linear Algebra with Application, 3rd Edition*" by Steven J. Leon.

1 Gram-Schmidt process

Let $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, the Q-factor of $\mathbf{A}$ be $\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, and the R-factor be

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}.$$

The Gram-Schmidt process of computing QR decomposition

1. $r_{11} = \|\mathbf{a}_1\| = 2.$[1] $\mathbf{q}_1 = \frac{1}{\|\mathbf{a}_1\|}\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$

2. $r_{12} = \mathbf{q}_1^T\mathbf{a}_2 = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} = 3$

[1] $||\cdot||$ is 2-norm in this example.

3. $\hat{\mathbf{q}}_2 = (\mathbf{I} - \mathbf{q}_1\mathbf{q}_1^T)\mathbf{a}_2 = \mathbf{a}_2 - \mathbf{q}_1\mathbf{q}_1^T\mathbf{a}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{pmatrix} =$
$\begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix}$

4. $r_{22} = \|\hat{\mathbf{q}}_2\| = 5$

5. $\mathbf{q}_2 = \dfrac{1}{\|\hat{\mathbf{q}}_2\|}\hat{\mathbf{q}}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$

6. $r_{13} = \mathbf{q}_1^T\mathbf{a}_3 = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix} = 2$

7. $r_{23} = \mathbf{q}_1^T\mathbf{a}_3 = \begin{pmatrix} -1/2 & 1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix} = -2$

8. $\hat{\mathbf{q}}_3 = (\mathbf{I} - \mathbf{q}_1\mathbf{q}_1^T)(\mathbf{I} - \mathbf{q}_2\mathbf{q}_2^T)\mathbf{a}_3 = (\mathbf{I} - \mathbf{q}_1\mathbf{q}_1^T - \mathbf{q}_2\mathbf{q}_2^T + \mathbf{q}_1\mathbf{q}_1^T\mathbf{q}_2\mathbf{q}_2^T)\mathbf{a}_3$

   Since $\mathbf{q}_1^T\mathbf{q}_2 = 0$,

   $\hat{\mathbf{q}}_3 = (\mathbf{I} - \mathbf{q}_1\mathbf{q}_1^T - \mathbf{q}_2\mathbf{q}_2^T)\mathbf{a}_3 = \mathbf{a}_3 - \mathbf{q}_1\mathbf{q}_1^T\mathbf{a}_3 - \mathbf{q}_2\mathbf{q}_2^T\mathbf{a}_3 = \mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2 =$
   $\begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix}$

9. $r_{33} = \|\hat{\mathbf{q}}_3\| = 4.$

10. $\mathbf{q}_3 = \frac{1}{\|\hat{\mathbf{q}}_3\|}\hat{\mathbf{q}}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$.

Therefore,

$$\mathbf{Q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}.$$

You can verify that $\mathbf{A} = \mathbf{QR}$ and $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$.

2 Householder transformation

Here we index $\mathbf{A}$ with superscript, and let $\mathbf{A}^{(0)} = \mathbf{A}$.

1. Compute the reflector $\mathbf{v}_1 = \mathbf{a}_1 - \text{sign}(a_{11})\|\mathbf{a}_1\|\mathbf{e}_1$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

2. Householder matrix $\mathbf{H}_1 = \mathbf{I} - 2\frac{\mathbf{v}_1\mathbf{v}_1^T}{\mathbf{v}_1^T\mathbf{v}_1}$.

$$\mathbf{H}_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

3. $\mathbf{H}_1\mathbf{A}^{(0)} = \mathbf{A}^{(0)} - \frac{1}{2}\mathbf{v}\mathbf{v}^T\mathbf{A}^{(0)} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & 8 & -4 \end{pmatrix}$

$$\mathbf{A}^{(1)} = \mathbf{H}_1\mathbf{A}^{(0)} = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -5 & 2 \end{pmatrix}$$

4. Now we only consider the submatrix $\mathbf{A}^{(1)}(2:4, 2:3) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \\ -5 & 2 \end{pmatrix}$ (using Matlab's notation), $\mathbf{A}^{(1)}(2:4, 2:3) = (\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)})$.

5. Let $\mathbf{v}_2 = \mathbf{a}_1^{(1)} + \text{sign}(\mathbf{A}^{(1)}(2,2))\|\mathbf{a}_1^{(1)}\|\mathbf{e}_1$

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}$$

6. $\mathbf{H}_2 = \begin{pmatrix} 1 & \\ & \mathbf{I} - 2\frac{\mathbf{v}_2\mathbf{v}_2^T}{\mathbf{v}_2^T\mathbf{v}_2} \end{pmatrix}$

$$\mathbf{I} - 2\frac{\mathbf{v}_2\mathbf{v}_2^T}{\mathbf{v}_2^T\mathbf{v}_2} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \frac{1}{25}\begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}\begin{pmatrix} -5 & 0 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

7. $\left(\mathbf{I} - 2\frac{\mathbf{v}_2\mathbf{v}_2^T}{\mathbf{v}_2^T\mathbf{v}_2}\right)\begin{pmatrix} 0 & 0 \\ 0 & 4 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \\ -5 & 2 \end{pmatrix} - \frac{1}{25}\begin{pmatrix} -5 \\ 0 \\ -5 \end{pmatrix}\begin{pmatrix} 25 & -10 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$

$$\mathbf{A}^{(2)} = \mathbf{H}_2\mathbf{A}^{(1)} = \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

8. The R-factor is $\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$, but where is the Q-factor?

9. Since $\mathbf{H}_2\mathbf{H}_1\mathbf{A} = \begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix}$,

$$\begin{aligned}
\mathbf{Q} &= (\mathbf{H}_2\mathbf{H}_1)^{-1} = (\mathbf{H}_2\mathbf{H}_1)^T = \mathbf{H}_1^T\mathbf{H}_2^T = \mathbf{H}_1\mathbf{H}_2. \\
&= \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}
\end{aligned}$$

3 Given's rotation

Let $\mathbf{A}^{(0)} = \mathbf{A} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}$.

1. Use a_{31} to eliminate a_{41}. $r_{3,4} = \sqrt{1^2 + 1^2} = \sqrt{2}$. $\begin{cases} \cos\theta_{3,4} = a_{31}/r = 1/\sqrt{2}, \\ \sin\theta_{3,4} = a_{41}/r = 1/\sqrt{2}. \end{cases}$

$$\mathbf{G}_{3,4}^{(1)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_{3,4} & \sin\theta_{3,4} \\ & & -\sin\theta_{3,4} & \cos\theta_{3,4} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\mathbf{A}^{(1)} = \mathbf{G}_{3,4}^{(1)}\mathbf{A}^{(0)} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ \sqrt{2} & 3/\sqrt{2} & \sqrt{2} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

2. Use a_{21} to eliminate a_{31}. $r_{2,3} = \sqrt{1^2 + \sqrt{2}^2} = \sqrt{3}$. $\begin{cases} \cos\theta_{2,3} = a_{21}/r = \sqrt{2}/\sqrt{3}, \\ \sin\theta_{2,3} = a_{31}/r = 1/\sqrt{3}. \end{cases}$

$$\mathbf{G}_{2,3}^{(1)} = \begin{pmatrix} 1 & & & \\ & \cos\theta_{2,3} & \sin\theta_{2,3} & \\ & -\sin\theta_{2,3} & \cos\theta_{2,3} & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & \sqrt{2}/\sqrt{3} & \\ & -\sqrt{2}/\sqrt{3} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^{(2)} &= \mathbf{G}_{2,3}^{(1)}\mathbf{A}^{(1)} = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & \sqrt{2}/\sqrt{3} & \\ & -\sqrt{2}/\sqrt{3} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ \sqrt{2} & 3/\sqrt{2} & \sqrt{2} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 4 \\ \sqrt{3} & 7/\sqrt{3} & 0 \\ 0 & -5/\sqrt{6} & \sqrt{6} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \end{aligned}$$

3. Use a_{11} to eliminate a_{21}. $r_{1,2} = \sqrt{1^2 + \sqrt{3}^2} = 2$. $\begin{cases} \cos\theta_{1,2} = a_{11}/r = 1/2, \\ \sin\theta_{1,2} = a_{21}/r = \sqrt{3}/2. \end{cases}$

$$\mathbf{G}_{1,2}^{(1)} = \begin{pmatrix} \cos\theta_{1,2} & \sin\theta_{1,2} & & \\ -\sin\theta_{1,2} & \cos\theta_{1,2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 & & \\ -\sqrt{3}/2 & 1/2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^{(3)} &= \mathbf{G}_{1,2}^{(1)}\mathbf{A}^{(2)} = \begin{pmatrix} 1/2 & \sqrt{3}/2 & & \\ -\sqrt{3}/2 & 1/2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ \sqrt{3} & 7/\sqrt{3} & 0 \\ 0 & -5/\sqrt{6} & \sqrt{6} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5/\sqrt{3} & -2\sqrt{3} \\ 0 & -5/\sqrt{6} & \sqrt{6} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \end{aligned}$$

4. Use a_{32} to eliminate a_{42}. $r_{3,4} = \sqrt{(-5/\sqrt{6})^2 + (-5/\sqrt{2})^2} = 10/\sqrt{6}$.
$$\begin{cases} \cos\theta_{3,4} = a_{32}/r = -1/2, \\ \sin\theta_{3,4} = a_{42}/r = -\sqrt{3}/2. \end{cases}$$

$$\mathbf{G}_{3,4}^{(2)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_{3,4} & \sin\theta_{3,4} \\ & & -\sin\theta_{3,4} & \cos\theta_{3,4} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1/2 & -\sqrt{3}/2 \\ & & \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^{(4)} &= \mathbf{G}_{3,4}^{(2)}\mathbf{A}^{(3)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1/2 & -\sqrt{3}/2 \\ & & \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5/\sqrt{3} & -2\sqrt{3} \\ 0 & -5/\sqrt{6} & \sqrt{6} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5/\sqrt{3} & -2\sqrt{3} \\ 0 & 10/\sqrt{6} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \end{aligned}$$

5. Use a_{22} to eliminate a_{32}. $r_{2,3} = \sqrt{(10/\sqrt{6})^2 + (5/\sqrt{3})^2} = 5$. $\begin{cases} \cos\theta_{2,3} = a_{22}/r = 1/\sqrt{3}, \\ \sin\theta_{2,3} = a_{32}/r = 2/\sqrt{6}. \end{cases}$

$$\mathbf{G}_{2,3}^{(2)} = \begin{pmatrix} 1 & & & \\ & \cos\theta_{2,3} & \sin\theta_{2,3} & \\ & -\sin\theta_{2,3} & \cos\theta_{2,3} & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & 2/\sqrt{6} & \\ & -2/\sqrt{6} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^{(5)} &= \mathbf{G}_{2,3}^{(2)}\mathbf{A}^{(4)} = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & 2/\sqrt{6} & \\ & -2/\sqrt{6} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5/\sqrt{3} & -2\sqrt{3} \\ 0 & 10/\sqrt{6} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \end{aligned}$$

6. Use a_{33} to eliminate a_{43}. $r_{3,4} = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$. $\begin{cases} \cos\theta_{3,4} = a_{33}/r = 1/\sqrt{2}, \\ \sin\theta_{3,4} = a_{43}/r = 1/\sqrt{2}. \end{cases}$

$$\mathbf{G}_{3,4}^{(3)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta_{3,4} & \sin\theta_{3,4} \\ & & -\sin\theta_{3,4} & \cos\theta_{3,4} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^{(6)} &= \mathbf{G}_{3,4}^{(3)}\mathbf{A}^{(5)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

7. The R-factor is $\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$; the Q-factor is

$$\begin{aligned} \mathbf{Q} &= \left(\mathbf{G}_{3,4}^{(3)}\mathbf{G}_{2,3}^{(2)}\mathbf{G}_{3,4}^{(2)}\mathbf{G}_{1,2}^{(1)}\mathbf{G}_{2,3}^{(1)}\mathbf{G}_{3,4}^{(1)}\right)^{-1} \\ &= \mathbf{G}_{3,4}^{(1)T}\mathbf{G}_{2,3}^{(1)T}\mathbf{G}_{1,2}^{(1)T}\mathbf{G}_{3,4}^{(2)T}\mathbf{G}_{2,3}^{(2)T}\mathbf{G}_{3,4}^{(3)T} \end{aligned}$$

$$\begin{aligned} \mathbf{G}_{2,3}^{(2)T}\mathbf{G}_{3,4}^{(3)T} &= \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & -2/\sqrt{6} & \\ & 2/\sqrt{6} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & -1/\sqrt{2} \\ & & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ & 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ & & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \end{aligned}$$

$$\mathbf{G}_{3,4}^{(2)T}\mathbf{G}_{2,3}^{(2)T}\mathbf{G}_{3,4}^{(3)T} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1/2 & \sqrt{3}/2 \\ & & -\sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ & 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ & & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$
= \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ & -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ & -1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}
$$

$$
\begin{aligned}
&\mathbf{G}_{1,2}^{(1)T}\mathbf{G}_{3,4}^{(2)T}\mathbf{G}_{2,3}^{(2)T}\mathbf{G}_{3,4}^{(3)T} \\
&= \begin{pmatrix} 1/2 & -\sqrt{3}/2 & & \\ \sqrt{3}/2 & 1/2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ & -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ & -1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ \sqrt{3}/2 & 1/2\sqrt{3} & -1/2\sqrt{3} & 1/2\sqrt{3} \\ & -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ & -1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{G}_{2,3}^{(1)T}\mathbf{G}_{1,2}^{(1)T}\mathbf{G}_{3,4}^{(2)T}\mathbf{G}_{2,3}^{(2)T}\mathbf{G}_{3,4}^{(3)T} \\
&= \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} & \\ & \sqrt{2}/\sqrt{3} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ \sqrt{3}/2 & 1/2\sqrt{3} & -1/2\sqrt{3} & 1/2\sqrt{3} \\ & -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ & -1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ & -1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{G}_{3,4}^{(1)T}\mathbf{G}_{2,3}^{(1)T}\mathbf{G}_{1,2}^{(1)T}\mathbf{G}_{3,4}^{(2)T}\mathbf{G}_{2,3}^{(2)T}\mathbf{G}_{3,4}^{(3)T} \\
&= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & -1/\sqrt{2} \\ & & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ & -1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix} = \mathbf{Q}
\end{aligned}
$$